

Cellular Automata: Some Source Material

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Publication history

This material is posted on the internet as a reference point for CSIRO (and other Australian and overseas) researchers in cellular automata, emphasising potential applications in complex systems science. It is not specifically a CSIRO scientific publication and it is inappropriate to cite it as such. Discussion of material in this report should reference the original sources, not this document. In particular, there is no commitment to archive this or any other version of this document beyond its period of immediate use, nor to locate it at any specific URL. The following information about versions is provided merely for the convenience of users: completeness is not guaranteed.

Version 0.5 Feb 2003. Discussion version for CSIRO CSS executive. Strong emphasis on Stochastic Cellular Automata. Circa 16 pages, over 40 references.

Version 1.0 First open release. March 4 2003. 25 pages, including code of Life and including CA figures and their postscript code.

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Comments, corrections and additions are welcome.

The CSIRO complex systems website is at:

[www.csiro.au/complex systems](http://www.csiro.au/complex_systems)

as an alias for:

www.ned.dem.csiro.au/BoschettiFabio/CSF/csiro-css.htm (interim)

Additional resources that are available are: the 'life' source code as listed in the appendix, the postscript files that form the basis for the figures, a bibtex file that includes all the references in this document and over 100 additional references in CSS. For these resources, check the websites (not yet available, March 2003), or otherwise contact the author.

1 Introduction

Cellular Automata have provided a powerful illustration of the emergence of complex behaviour from simple specifications. Cellular Automata have also been used to model a number of specific natural systems.

A Cellular Automaton is defined as a system defined at discrete time steps with a discrete spatial geometry (generally a regular lattice). The state of the system is defined in terms of discrete variables at each point in time. The Cellular Automaton is specified in terms of rules that define how the state changes from one time to the next. These rules for evolution from time t to $t + 1$ are specified as functions of the states of a ‘small’ number of sites at time t .

In specifying Cellular Automata, some notation is in order:

- q is used to denote the number of states per site; the states of sites are specified as integers, n , (subscripted as required to identify location and/or time) taking values 0 to $q - 1$;
- D is used to denote the spatial dimension. Generally, a Cellular Automaton is defined on a D -dimensional hypercubic lattice, i.e. a linear chain for $D = 1$, square lattice for $D = 2$, etc.
- r is the range of influence on a hypercubic lattice, i.e. the set of sites that are arguments of the function defining the evolution rule. Thus $r = 1$ implies nearest-neighbour influence.
- M denotes the number of sites that influence the evolution of each site. For a site, k , the set of sites influencing its evolution is denoted $B(k)$. Thus $M = |B(k)|$. In D dimensions, a range of r implies influence from $M \leq (2r + 1)^D$ sites. The M sites can jointly take q^M states and each state leads to one of the q states for the updated site. Thus a total of $q^{(q^M)}$ distinct rules are possible.
- N is used to denote the number of sites in a finite Cellular Automaton. Therefore an N -site Cellular Automaton can only have q^N different states and its behaviour must be cyclic with a period $\leq q^N$. More generally, we regard the lattice sites as forming a set A so that if A is finite, then $N = |A|$.

We follow Wolfram in using a standard indexing for Cellular Automata rules. Each rule is given an integer identifier from the range 0 to $q^{(q^M)} - 1$. These are regarded as being a base- q representation in terms of q^M digits. The position of the digit represents the initial state of the M sites (again as a base- q representation) and the value of the digit represents the updated state. Usually, the rules are expressed in decimal notation. Of course, this type of specification is only meaningful for specified values of q and M with a specified ordering of the M sites. It is mainly used for $D = 1$ (whence $M = 2r + 1$). In this document, rules cited without specification refer to $q = 2, r = 1, D = 1$. Other cases are specified as, for example, $33_{[q=3,r=2]}$ (with $D = 1$ implicit) or $67_{[q=3,r=2,D=2]}$.

This document is intended to evolve to meet the needs of the CSIRO CSS initiative. The coverage is:

- The reprint volume *Theory and Applications of Cellular Automata* edited by Wolfram [47] includes a number of key papers (pre-1986). The present document cites a number

(maybe most/all later) of these. The volume also includes an annotated bibliography. A few of these are cited. The volume also includes (as appendices) tables of properties of one-dimensional Cellular Automata, primarily for $q = 2, r = 1$.

- The coverage of Stochastic Cellular Automata mostly reflects early work related to lattice statistical mechanics.
- The volume *A New Kind of Science* [48] is notable for, among other things, its limited citation of other work and its considerable length. To aid the reader, references to points of interest in that book generally include the page number.

Workers interested in following up on particular aspects can, of course, also look for works that cite references that are included here.

2 Why study Cellular Automata?

2.1 Overview

Some of the generic reasons for studying Cellular Automata are:

- to study generic aspects of complex systems;
- to provide simple test cases for techniques designed to analyse complex systems;
- to provide (in cases where universality applies) quantitative information about complex systems.

Cellular Automata also have the advantage of allowing exact calculation, thereby removing concerns that results are artifacts of numerical procedures.

The role of Cellular Automaton models for understanding generic aspects of complexity has been emphasised by Wolfram, most notably [41, 42, 48].

Cellular Automata have also been used to model a number of specific complex systems.

2.2 Computational measures of complexity

One approach to describing the complexity of particular Cellular Automata is in terms of the complexity of the computer programs that are required to describe their behaviour. The complexity is taken as low if there is some sort of reduced description of the behaviour.

Part of the significance of this is the following argument:

- there are problems that cannot necessarily be answered by any general purpose computer (e.g. analysing whether a particular computer program will stop in a finite time);
- some cellular automata can be set up to emulate such general purpose computers;
- therefore there are questions about cellular automata that cannot be answered by any general purpose computer.

3 Results for Cellular Automata

3.1 Early

An early exposition of Cellular Automata, was given by Wolfram [41]. This concentrated on the simplest case, $q = 2, r = 1, D = 1$. This allows 256 rules, but Wolfram restricted consideration to the 32 rules (termed ‘legal’ rules) which satisfied two additional constraints:

- invariance of the ‘null’ (all zero) state;
- symmetry under spatial reflection.

A subsequent analysis by Wolfram [43] produced the fourfold classification of behaviour:

1. Cellular Automata that evolve to invariant states;
2. Cellular Automata that evolve to periodic states;
3. Cellular Automata that evolve to chaotic states;
4. Cellular Automata whose evolution can produce persistent structures.

The first three types of behaviour had been identified (in $q = 2, r = 1, D = 1$) by Wolfram [41]. Case 4 was identified in case with $q = 2, r = 2, D = 1$. However, if the restriction of requiring reflection symmetry is dropped, then persistent structures can occur with $q = 2, r = 1, D = 1$ as in rule 110. Figure 1 shows examples of cases 2, 3 and 4.

The value of this classification has been queried. For the present we note:

- Wolfram [48, p240] notes that there are intermediate classes that fall between cases identified above, but indicates that they are rare;
- class 3 includes both the rules that produce apparent randomness (e.g. rule 30, see below) and those that produce regular fractal patterns.

Some additional important early results are:

- analysis of the actual lengths of cycles for finite Cellular Automata with additive rules for $D = 1, r = 1, q = 2, 3, 4$ and $N \leq 40$ [25].

3.2 A New Kind of Science

Wolfram’s book, *A New Kind of Science*, [48] gives an extensive account of Cellular Automata and makes a number of sweeping claims (starting with the title) for the importance of this work.

One of the interesting concepts is a classification of randomness into three sources, identified [48, Chapter 7] as:

- randomness from the environment;
- randomness from initial conditions;
- intrinsically generated randomness.

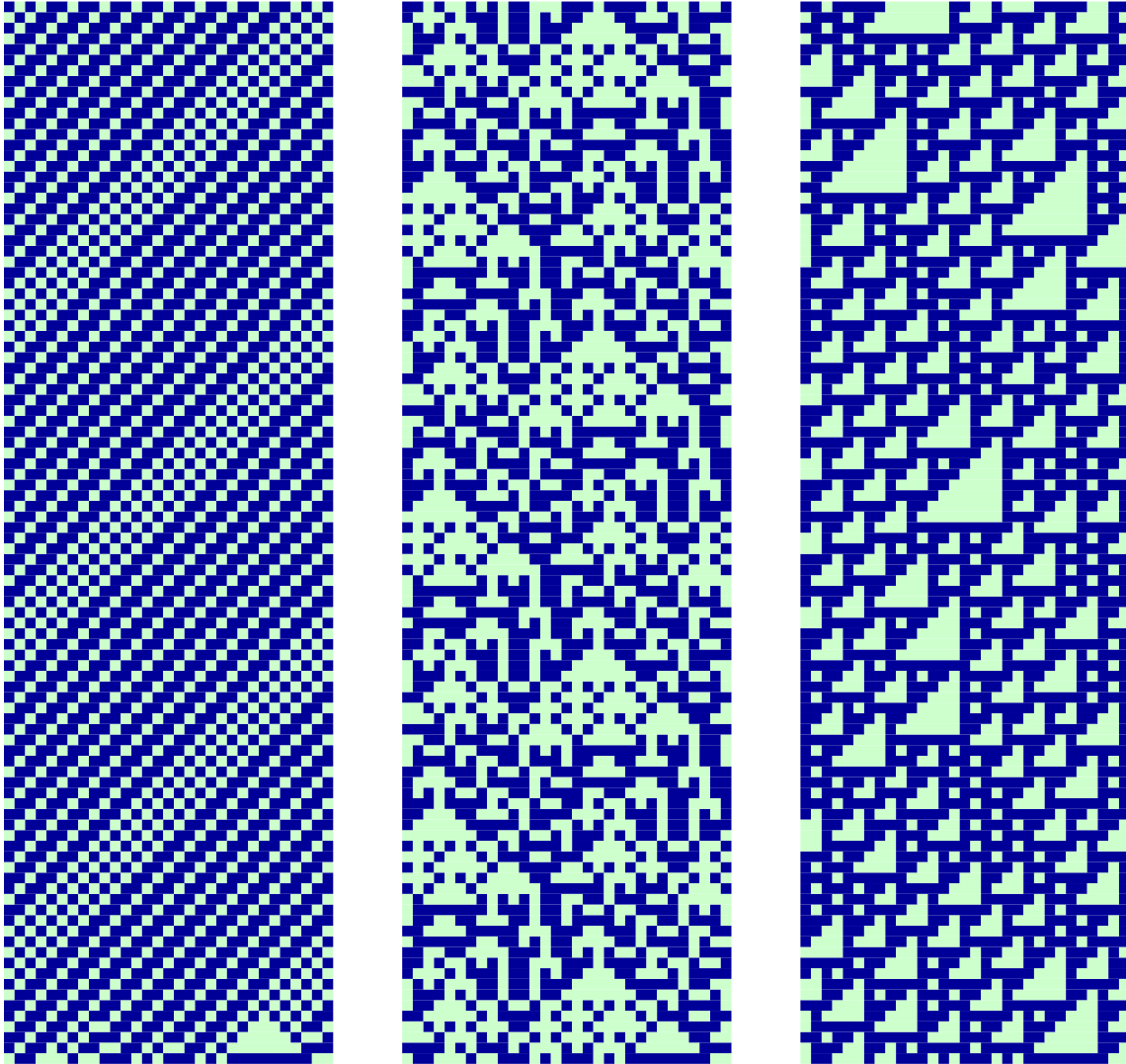


Figure 1: Cellular automata evolving from disordered states, for cellular automata 58, 90 and 110. Time increases up the page. The postscript source code for this figure is given as an appendix.

In particular, *Mathematica* uses Cellular Automaton number 30 to generate random binary digits, with the specific application and general principle being subject to US patent 4,691,291 [48, p973]. The generation of randomness from simple rules was discussed by Wolfram [44, 45] who considered both rule 30:

$$n'_k = (n_{k-1} + n_k + n_{k+1} + n_k n_{k+1}) \pmod{2}$$

and also rule 45:

$$n'_k = (1 + n_{k-1} + n_{k+1} + n_k n_{k+1}) \pmod{2}$$

(noting that these rules are equivalent, under spatial reflection) to rules 86 and 75 respectively).

However, perhaps the most significant result reported in *A New Kind of Science* is that Cellular Automaton 110 is capable of universal computation, i.e. with appropriate initial conditions it can, in principle, be set up emulate any general purpose computer.

3.3 A Needless Kind of Science

The publication of *A New Kind of Science* [48] attracted a significant amount of unfavourable comment. One contributing reason is identified in the notes [48, p849] *... I have chosen to explain straightforwardly the importance I believe my various results have. Perhaps I might avoid some criticism by a greater display of modesty, but the cost would be a drastic reduction in clarity.*

However, the lack of modesty is apparent not only in the claims of importance, but also in the minimal of mention of work by anyone other than the author. Through the book there seems to be a shift in language. Early chapters (and associated notes) refer to ‘*the discoveries in this book* while later chapters (and their notes) generally refer to ‘*my discoveries*’.

However, the main disappointment in reading *A New Kind of Science* is how little of the contents are new.

4 Self-Organised Criticality

One system, closely related to Cellular Automata, is the sand-pile model [2, 3], which was developed as a notional model for describing avalanche sizes. It is generally defined in terms of sites on a square lattice whose states are incremented (‘addition of sand’) at a slow rate until one site reaches a predefined stability limit. At this point, its ‘sand’ is distributed across neighbour sites and (within the same time-step) they redistribute their ‘sand’ if distributions from other sites causes them to exceed the stability limit. The most striking property of this model is that it exhibits *Self-Organised Criticality* (SOC) in which power-law behaviour appears spontaneously.

The study of self-organised criticality experienced an explosive growth following the publication of the sand-pile model [2, 3]. This work was cited in thousands of subsequent papers. Overviews are given in books by Bak [1] (making some sweeping claims of applicability, again starting with the title: *How Nature Works: ...*) and Buchanan [7]. A key motivation is that while Mandelbrot [24] (and much following work) identified the wide-spread occurrence of fractal properties in natural and human systems, this identification is essentially descriptive in

that it identifies the geometrical properties. The significance of the sand-pile model and its generalisations is that it appears to provide the possibility of a mechanistic basis for the appearance of fractal properties.

In thermodynamics, a critical point is a special point. In liquid gas systems it occurs only when temperature and pressure are **both** at specific values, T_c and P_c respectively, (e.g. 373.99°C and 22.066 MPa for water). The special nature of the critical point reflects the structure of the phase space. An isolated critical point is the end-point of a 2-phase co-existence line — a line of critical points is the boundary of a 2-phase co-existence surface etc. In Self-Organised Criticality, the power law behaviour appears apparently without any need to set such specific control variables. The account of Self-Organised Criticality by Sornette [31, chapter 15] suggests that it is the separation of scales (the fast scale of the avalanches vs. the slow scale of adding ‘sand’) that creates the pre-conditions for Self-Organised Criticality.

5 Stochastic Cellular Automata

5.1 Motivation

In *A New Kind of Science* [48, p922], Wolfram suggests (with little further comment) that the behaviour of Stochastic Cellular Automata is generally similar to that of Continuous Cellular Automata.

However, some of the reasons for studying Stochastic Cellular Automata in their own right are:

- they should provide greater modelling flexibility than Deterministic Cellular Automata. Although Deterministic Cellular Automata can produce random behaviour (and thus mimic Stochastic Cellular Automata — see for example Creutz [8] as cited below) the added complexity and reduced transparency when using Deterministic Cellular Automata in this way may make direct use of Stochastic Cellular Automata preferable.
- There is the scope for analysing these Cellular Automata using the standard armoury of tools from statistical mechanics — series expansion, renormalisation group etc. In particular the multiple perspectives described below increase the number of ways in which techniques from statistical mechanics might be applied.
- Stochastic Cellular Automata interpolate between Deterministic Cellular Automata and can thus provide a framework for exploring the ‘rule space’ of Cellular Automata.

The reprint volume [47] includes several papers on Stochastic Cellular Automata: [21, 17, 20, 9, 18, 5]. The field of Stochastic Cellular Automata was reviewed by Rujan [30].

As generalisations of Deterministic Cellular Automata we require Stochastic Cellular Automata to have their evolution determined by sites over a small range, We express this as requiring:

$$P(\{n_{k,t+1} : k \in A\} | \{n_{k,t} : k \in A\}) = \prod_{k \in A} \Pr(n_{k,t+1} | \{n_{j,t} : j \in B(k)\})$$

5.2 Multi-perspectives

The starting point for considering multiple perspectives on Stochastic Cellular Automata is the Gibbs formalism of statistical mechanics. For a system of N q -state variables, n_k $k = 1, N$, this expresses the probability for each of the q^N micro-states of the system in terms of an energy $E(\{n_k\})$. The probability of the micro-states takes the form

$$\Pr(\{n_k : k \in A\}) \propto \exp[-E(\{n_k\})/k_B T]$$

where T is the temperature and k_B is Boltzmann's constant. Generally, the energy is in the form of a sum of contributions that each involve only a small number of neighbouring sites. Writing this as

$$E = \sum_{k \in A} E_k(n_k, \{n_j : j \in B(k)\})$$

implies

$$\Pr(\{n_k : k \in A\}) \propto \prod_{k \in A} \exp[-E_k(n_k, \{n_j : j \in B(k)\})/k_B T]$$

The various different perspectives that can be applied to Stochastic Cellular Automata are:

- 1 One can, through the standard techniques of statistical mechanics, analyse $\Pr(\{n_{k,t}\})$, for a particular time t , generally in terms of various moments, by considering it as a D -dimensional random field.
- 2 One of the standard ways of analysing statistical mechanics systems is by Monte Carlo simulation. This is a technique of importance sampling that generates a sequence of microstates with probabilities given by the Gibbs distribution. The sequence of states is created by a sequence of local transformations [6]. However the process can also correspond to a $D + 1$ -dimensional Stochastic Cellular Automaton.
- 3 However, the time-evolution described above can be given a specific interpretation as a physical model of the time evolution of a D -dimensional system. As such one can consider the scaling behaviour in space as well as time. This is the field of dynamic critical phenomena [19]. One important result from the theory of dynamic critical behaviour is that the imposition of conservation laws can change the universality class of the temporal critical exponents.
- 4 The final perspective is to realise that, for discrete time-evolution one has a set of values $\{n_{k,t}\}$ with k from a D -dimensional lattice and t discrete in time, thus defining a random field in $D + 1$ dimensions. One can then apply the standard statistical mechanics techniques for analysing the $D + 1$ -dimensional field.

The example in Figure 2 gives concrete examples of some of these different ways of looking at Stochastic Cellular Automata.

Mapping the $D + 1$ -dimensional distribution of a Stochastic Cellular Automaton onto a Gibbs field requires equating the probability distributions $\Pr(\{n_{k,t}\})$. Writing the Stochastic Cellular Automaton distribution as a product over conditional distributions for single time-steps as

$$\Pr(\{n_{k,t}\}) = \prod_t \Pr(\{n_{k,t}\}|\{n_{k,t-1}\})$$

and then noting that these are expressed as products over local stochastic rules, leads to a requirement to equate local expressions:

$$\Pr(n_{k,t+1}|\{n_{j,t} : j \in B(k)\}) = \exp[-E_k(n_{k,t+1}, \{n_{j,t} : j \in B(k)\})/k_B T]$$

Two important results are:

- For any Stochastic Cellular Automaton, an energy expression can be found, i.e. the Stochastic Cellular Automaton distributions are a subset of the Gibbs distributions;
- Overlaps in the interaction sets $B(k)$ for different sites can lead to a choice in which interactions contribute to which of the E_k , i.e. in solving for the E_k given the Stochastic Cellular Automaton rules, some interactions may be split between two or more of the E_k .

5.3 Disorder points

Enting [unpublished D. Sc. thesis, Monash University] noted three strands of research, initially separate, that coincidentally appeared in a brief flurry of activity in Canberra in the late 1970s. These applied mainly to binary (i.e. $q = 2$) systems and in many cases the work was expressed in terms of ‘spin’ variables taking values ± 1 .

1. Models were constructed to have soluble spatial statistics through constructions that reproduced 1-D Markov chains over 2-D lattices [34, 35]. This was generalised, with a view to applications in stochastic spatial modelling, by Pickard [27, 28, 29].
2. Stochastic Cellular Automata were used to simulate the growth of disordered mixed crystals in order to determine the extent to which X-ray diffraction data could be used to recover the spatial statistics of such crystals [38, 39, 36, 14, 37].
3. The type of transformation described at the end of the previous section was used to study the relationship between 1+1 dimensional Stochastic Cellular Automata and 2-D generalised Ising models [10, 11, 12, 13, 14].

The various cases considered several different topologies of lattice.

Some of the key results were:

- Stochastic Cellular Automata can be mapped onto Gibbs distributions (as described above) and this can identify symmetries and give new solutions
- Even when restricted to symmetric cases, a specification of the two-site correlation functions was insufficient to uniquely characterise a Stochastic Cellular Automaton. In terms of interpreting X-ray data (which give essentially the Fourier transform of the 2-site correlation), this means that X-ray data are insufficient to fully define the structure of mixed disordered crystals.
- A number of special cases of reversible Stochastic Cellular Automata cases corresponded to Gibbs distributions in which particularly simple correlations structures occurred. These special cases had previously been identified from more general exact solutions by Stephenson [32, 33] who termed the special cases ‘disorder points’. This term has been widely adopted for many of the special cases of Gibbs distributions that correspond to reversible Stochastic Cellular Automata.

If the Gibbs distribution is symmetric under reversal of the time direction then the decomposition can be written as

$$\Pr(\{n_{k,t} : k \in A, t \in [-\tau, \tau]\}) = \prod_{t < 0} \Pr(\{n_{k,t}\} | \{n_{k,t+1}\}) \prod_{t > 0} \Pr(\{n_{k,t}\} | \{n_{k,t-1}\}) \phi(\{n_{k,t=0}\})$$

In this form, the distribution of the sites $n_{k,t=0}$ is independent of all the other sites and so

$$\Pr(\{n_{k,t=0} : k \in A\}) = \phi(\{n_{k,t=0}\})$$

As with the multilayer distributions, $\phi(\{n_{k,t=0}\})$ will be a product of local factors, thus corresponding to an easily-soluble short-range 1-dimensional lattice statistics model. This is the form of the special solutions for correlations found by Stephenson. The general form of extracting a 1-dimensional distribution in reversible Stochastic Cellular Automata was described by Enting [13]. A particular case of this type of transformation, had been found previously for the anisotropic triangular Ising model by Gibberd [16].

6 Two spatial dimensions

6.1 Deterministic

The study of Cellular Automata achieved a high profile with the so-called ‘Game of Life’ developed by John Conway to illustrate growth of organisms under competing pressures of overcrowding vs. too few neighbours. The behaviour is notable for complex persistent structures. This model has been shown to be capable of supporting universal computation.

A review of 2-D Cellular Automata was given by Packard and Wolfram [26].

Two dimensional Cellular Automata have also been used extensively in image-processing applications. This is generally outside the scope of complex systems science.

More recently, the sand-pile model [2, 3], as an example of Self-Organised Criticality, has almost certainly become the most-widely studied Cellular Automaton.

6.2 Stochastic

As noted above, simplified solutions can be found for a $D+1$ dimensional Stochastic Cellular Automaton whose equivalent Gibbs distribution in $D + 1$ dimensions is symmetric under time reversal. Writing the Gibbs distribution as two products of Stochastic Cellular Automaton probabilities (for increasing and decreasing times) leaves a residual factor that gives the probability of the D -dimensional distribution of $\{n_{k,t=0}\}$. This approach is not restricted to the $D = 1$ cases noted above.

Enting and Welberry [14] considered a Stochastic Cellular Automaton defined as successive triangular layers (with the [111] orientation) on a simple cubic lattice in 3 dimensions with time in the direction of the cube diagonal). The residual distribution on the triangular layers was that of the triangular lattice Ising model and so a phase transition could occur for appropriate values of the probabilities. This is different from Stochastic Cellular Automata in 1+1 dimensions since the residual distributions correspond to 1-dimensional Gibbs field, in which phase transitions only occur at the zero-temperature limit. Welberry and Miller [40] describe the implications of this class of solutions for crystallographic studies.

The same principle can be applied to other 3-D lattices which comprise successive triangular layers — most notably the face-centred-cubic and the hexagonal-close-packed lattices.

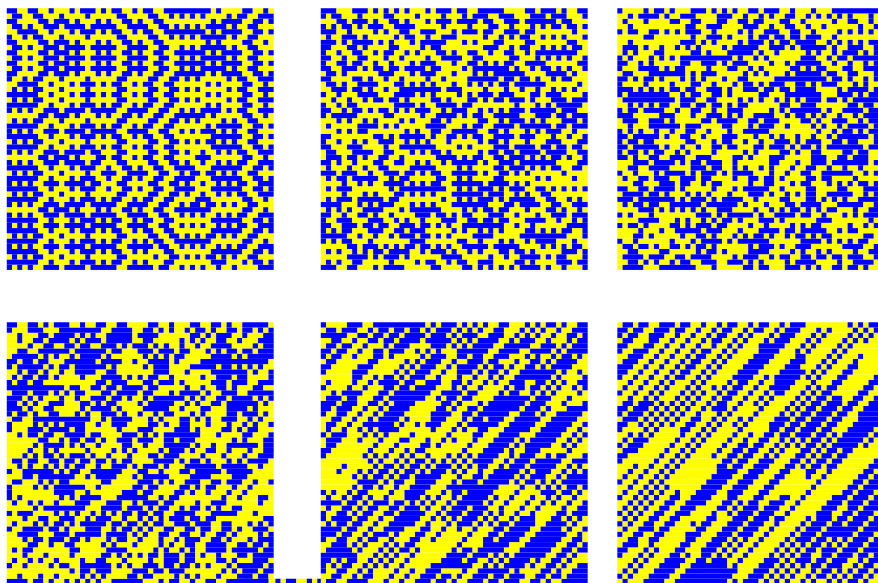


Figure 2: Stochastic cellular automata (with time evolving in the upwards/rightwards diagonal direction) interpolating between a case with alternating structure and an ‘identity’ case. This illustrates some of the different perspectives described in the previous section: in the spatial direction, the values are actually uncorrelated (i.e. a 0-dimensional distribution, rather than the more common case of a 1-dimensional distribution): the actual construction used the 1+1 dimensional Stochastic Cellular Automaton formalism; the 2-dimensional distribution is given by a statistical mechanics model with 4-site interactions and two-site interactions along the diagonals of each unit square.

7 Related Models

There are a wide range of models that are related to Cellular Automata, particularly Stochastic Cellular Automata where ‘related models’ potentially includes all of lattice statistical mechanics.

Models that are closely related include:

- directed percolation, see for example [31, Section 12.4].

Concluding remarks

A summary such as this must, of course, be highly selective in its coverage. The starting point largely reflects the author’s interests in Stochastic Cellular Automaton. The aim is for the coverage to extend to reflect the breadth of CSIRO (and other Australian) research in the application of Cellular Automata and related models of complex systems.

Topics deserving additional coverage include:

- Pickard’s work on reproducing 1-dimensional Markov chains across a 2-dimensional lattice [27, 28, 29].
- Searching the rule-space for rules with desired behaviour.
- Langton’s λ parameter description – is this related to statistical closures?
- Measures of complexity of Cellular Automata.
- Other work on disorder points.

Application areas include:

- hydrodynamics.

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Appendix A: Notation

A The set of spatial sites in a Cellular Automaton.

$B(k)$ The set of spatial sites that define the single-step evolution of site k in a Cellular Automaton.

D Spatial dimensionality.

j, k Spatial site indices in Cellular Automata.

k_B Boltzmann's constant.

M Number of sites in range of interactions, i.e. $M = |B(k)|$. Consequently, $M \leq (2r + 1)^D$ on a D -dimensional hypercubic lattice.

n, n_k State variable (at site k) — an integer in range 0 to $q - 1$.

N Number of sites in a finite Cellular Automaton, i.e. $N = |A|$.

q Number of states per site.

r Range of interaction, in units of the lattice constant.

t Time, generally taking integer values.

T Temperature in thermodynamic systems.

Appendix B: Glossary

critical exponent The exponent in a power law relation describing scaling behaviour at or around a critical point.

critical point In thermodynamics, the end-point of a 2-phase co-existence line.

disorder point A point at which competing interactions in a lattice system cancel to give a behaviour characteristic of a lower spatial dimensionality. Note that (except in special cases) this is **not** the point of transition between ordered and disordered phases.

fractal [noun/adjective] An object (or property of such an object) showing geometric self-similarity under change of scale [24].

Gibbs distribution The probability distribution for micro-states of a physical system, expressed as $\text{Pr}(\{n_k\}) \propto \exp[-E(\{n_k\})/k_B T]$

Gibbs random field A random field specified by the Gibbs distribution.

Ising model A model of magnetic phase transitions (and other analogous transitions) in statistical mechanics.

life A 2-dimensional Cellular Automaton notionally representing the birth and death of groups of organisms in an environment that penalises both isolation and overcrowding. The behaviour is notable for complex persistent structures. This model has been shown to be capable of supporting universal computation.

Markov random field A random field defined in terms of local conditional probabilities.

random field A statistical model of variables distributed in space (generally on a regular lattice) described in terms of some probability distribution.

renormalisation group A class of computational techniques that determine critical behaviour from calculations of how the system interactions transform into equivalent interactions at larger scales, see for example [15].

scaling A description of the way in which different the scales of physical properties in systems are related to each other. In systems whose properties lack any characteristic sizes, these relations are in terms of power laws. Scaling relations define connections between exponents.

Self-Organised Criticality The property of showing power law behaviour without tuning the system parameters to lie at the end of a 2-phase co-existence line.

statistical mechanics The branch of theoretical physics devoted to determining bulk thermodynamic quantities from underlying atomic/molecular scale interactions, generally using the Gibbs distribution as a starting point.

universality It is found that in many cases, systems at criticality exhibit the same scaling behaviour and in particular have the same set of critical exponents. In simple cases these exponents depend on the spatial dimension of the system and the symmetry of the interacting variables. The classes of equivalent systems are called universality classes.

Appendix C: Some Internet resources

Complex systems A valuable resource is the virtual library in complex systems at Charles Sturt University:

http://lorenz.mur.csu.edu.au/vl_complex/topics.html

This contains entries for topics including cellular automata, the ‘life’ Cellular Automaton, percolation etc.

ANKOS The official website for the book *A New Kind of Science* [48] is:

<http://www.wolframscience.com>

Appendix D: The Wolfram questions

In 1985, Wolfram [46, reprinted in [47]] posed a list of twenty problems in the theory of Cellular Automata. Additional discussion explained the significance of the problems. The questions were:

1. *What overall classification of Cellular Automata can be given?*
2. *What are the exact relations between entropies and Lyapunov exponents for Cellular Automata?*
3. *What is the analogue of geometry of configuration space of Cellular Automata?*
4. *What statistical quantities characterize Cellular Automaton behaviour?*
5. *What invariants are there in Cellular Automaton evolution?*
6. *How does thermodynamics apply to Cellular Automata?*
7. *How is different behaviour distributed in the space of Cellular Automaton rules?*
8. *What are the scaling properties of Cellular Automata?*
9. *What is the correspondence between Cellular Automata and continuous systems?*
10. *What is the correspondence between Cellular Automata and stochastic systems?*
11. *How are Cellular Automata affected by noise and other imperfections?*
12. *Is regular language complexity generically non-decreasing with time in one-dimensional Cellular Automata?*
13. *What limit sets can Cellular Automata produce?*
14. *What are the connections between the computational and statistical characteristics of Cellular Automata?*
15. *How random are the sequences generated by Cellular Automata?*
16. *How common are computational universality and undecidability in Cellular Automata?*
17. *What is the nature of the infinite size limit of Cellular Automata?*
18. *How common is computational irreducibility in Cellular Automata?*
19. *How common are computationally intractable problems about Cellular Automata?*
20. *What higher-level description of information processing in Cellular Automata can be given?*

Techniques of statistical mechanics

There are a number of classes of technique that have been developed for investigating cooperative phase transitions:

exact solutions In some very special cases, exact solutions of non-trivial lattice statistics problems are possible [4].

power series expansions These expand the thermodynamic quantities as power series, almost always expanding about either a fully-ordered state (zero temperature) or a completely random state (infinite temperature) [23, Chapter 7].

statistical closures These involve averaging aspects of the statistical distributions of the microstates. At least in low dimensions (2 or 3) they frequently give incorrect values of the power-law exponents [22].

renormalisation group Renormalisation group techniques operate by defining the way in which effective interaction probabilities change under change of length scale. The power-law behaviour arises from the limiting (fixed-point) characteristics of these transformation [15].

Monte Carlo simulations These act to simulate the statistical distributions by working directly from the microstate probabilities [6].

Code for life

The following listing gives a program in Borland Turbo Pascal to implement the 'Life' Cellular Automaton. The choice of language is mainly because of the ability (under the DOS operating system, or from a DOS window in various 32-bit Windows OS) to write to single screen pixels (or some emulation thereof). The point of providing this code is that it can be readily modified. Note that the graphics call in the main program has to specify the correct path for the system graphics (BGI) files. Turbo Pascal has now been superseded by Delphi and versions of Turbo Pascal are now available free.

```
program life;
{$R+}
{Simulation of Conway's 'Life' Cellular Automaton}
{ I. Enting, 1999}
{Last update 11 Feb 2003}
uses graph;
const n= 121;
      n1 = n-1;
      q = 2;
      q1 = q-1;
var mm, vv, jj, j3,j4: integer;
    phase : integer; {Alternate between 0 and 1
                      as 3rd index of aa[ ] }
    aa : array[0..n1,0..n1,0..1] of integer;
    cc, count : array[0..q1] of integer;
    xx : array[0.. 10] of char; {For output of patterns}
    fout :text;

procedure dopixel(i,j,k: integer);
{
  *** Write to screen and update 'new' array ***
}
begin
  putpixel(3*i+10,2*j+10,cc[k]);
  aa[i,j,1-phase] := k;
end;

procedure update;
{
  *** Scan through lattice and apply CA rule ***
}
var i,j,k,m, m1, i1, j1,ii,jj,cc, iz,jz : integer;
    ss : array[0..q1] of real; st,kk :real;
begin
  for jz := 0 to n1 do
    for iz := 0 to n1 do
```

```

begin
  cc := -aa[iz,jz,phase];
  for jj := jz-1 to jz +1 do
  for ii := iz -1 to iz+1 do
    begin
      i1 := (n+ii) mod n;
      j1 := (n+jj) mod n;
      cc := cc+ aa[i1,j1,phase];
    end;
  k := 0;
  if( cc = 2) then k := aa[iz,jz,phase];
  if( cc = 3) then k :=1;
  dopixel(iz,jz,k);
end;
phase := 1-phase;
end;

```

```

procedure initran;
var i,j, k : integer; v: real;
begin
  for i := 0 to n1 do
    for j := 0 to n1 do
      begin
        k := random(2);
        if ( k > 1) then k:= 1;
        dopixel(i,j,k);
      end;
  phase := 1 -phase;
end;

```

```

procedure outa;
{ Capability of writing out patterns}
var i,j,k : integer;
begin
  assign(fout,'life.dat');
  rewrite(fout);
  write(fout,'% Life simulation');
k := 0;
for j := 0 to n1 do
  begin
  for i := 0 to n1 do
    begin
      write(fout, ' ',xx[aa[i,j,phase]]);
      k := k+1;
      if( k > 30 ) then

```

```

        begin writeln(fout); k := 0; end;
    end;
write(fout,' lf ');
end;
writeln(fout);
writeln(fout,' showpage');
close(fout);
end;

begin
    cc[0] := 0; cc[1] := 14; {cc[2] := 9;}
    xx[0] := 'A'; xx[1] := 'B'; xx[2] := 'C';
    writeln('.....');
    mm := 0;
    initgraph(mm,vv,'d:\tp\bgi'); {needs correct path}
    setgraphmode(1);
    phase := 0;
    initran;
    jj := 0;
    while jj >= 0 do
        begin
            for j3 := 1 to 100 do update;
                readln(jj); {Pause and wait for integer input}
            end;
        closegraph;
        { outa;}
    end.

```

Code for figure

```
%!PS-Adobe-3.0 EPSF-3.0
%%BoundingBox: 90 90 540 510
%%Title: File cas.eps
%%CreationDate: Last change 13/2/03
%
% Plot 2-state cellular automata
%
%
/xdif { exch def} def
/col1 {0.8 1 0.8 setrgbcolor} def
/col2 {0 0 0.6 setrgbcolor} def
/col3 {0 0 1 setrgbcolor} def
/dx 4 def
dx setlinewidth
/A { col1 block} def
/B { col2 block} def
/C { col3 block} def
/LF { /xc x0 def yc dx add /yc exch def} def
/block {newpath xc yc moveto dx 0 rlineto stroke
        xc dx add /xc exch def} def

/yc 100 def
/pcase { 0 eq {A} {B} ifelse } def

/nsize 30 def % matrix size
/nlim nsize 1 sub def
/nc 0 def

/vvstore {3 dict begin /vn xdef /vc xdef /vp xdef } def
/not { 1 sub neg } def

% vp, vc, vn func vnew
%/func { add add 2 mod } def
/func90 { exch pop add 2 mod } def % RULE 90
/func58 { vvstore vn vc not mul vn not vp mul add end} def
/func110 { 3 copy 1 add mul mul neg
          exch pop add add } def % RULE 110?

%
%array n nget val
/nget { nsize add nsize mod get} def
%
% array Print
```

```

/Print { {pcase} forall LF} def

/ycalc {/nc exch def
    nc 1 sub xx exch nget
    nc xx exch nget
    nc 1 add xx exch nget
    func yy exch nc exch put } def
%
/xcalc {/nc exch def
    nc 1 sub yy exch nget
    nc yy exch nget
    nc 1 add yy exch nget
    func xx exch nc exch put } def

/yc 100 def
/x0 100 def
/xc x0 def

/func {func58} def
% starting config: 2nd last digits Potter, M in phone book
/xx [ 0 1 1 0 0 0 1 0 0 1 1 1 0 1 0 1 1 1
    0 1 0 1 1 1 1 1 1 1 0 0] def
% yy is just a paceholder
/yy [ 0 1 1 0 0 0 1 0 0 1 1 1 0 1 0 1 1 1
    0 1 0 1 1 1 1 1 1 1 0 0] def
/Loop { xx Print
    0 1 nsize {ycalc} for
    yy Print
    0 1 nsize {xcalc} for } def
1 1 50 {Loop} for
%
/yc 100 def
/x0 250 def
/xc x0 def

/func {func90} def
% starting config: 2nd last digits Potter, M in phone book
/xx [ 0 1 1 0 0 0 1 0 0 1 1 1 0 1 0 1 1 1
    0 1 0 1 1 1 1 1 1 1 0 0] def
% yy is just a paceholder
/yy [ 0 1 1 0 0 0 1 0 0 1 1 1 0 1 0 1 1 1
    0 1 0 1 1 1 1 1 1 1 0 0] def
/Loop { xx Print
    0 1 nsize {ycalc} for
    yy Print

```

```

        0 1 nsize {xcalc} for } def
1 1 50 {Loop} for
%
/yc 100 def
/x0 400 def
/xc x0 def

/func {func110} def
% starting config: 2nd last digits Potter, M in phone book
/xx [ 0 1 1 0 0 0 1 0 0 1 1 1 0 1 0 1 1 1
    0 1 0 1 1 1 1 1 1 1 0 0] def
% yy is just a paceholder
/yy [ 0 1 1 0 0 0 1 0 0 1 1 1 0 1 0 1 1 1
    0 1 0 1 1 1 1 1 1 1 0 0] def
/Loop { xx Print
        0 1 nsize {ycalc} for
        yy Print
        0 1 nsize {xcalc} for } def
1 1 50 {Loop} for
%
showpage

```